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Introduction to nonlinear finite element modeling

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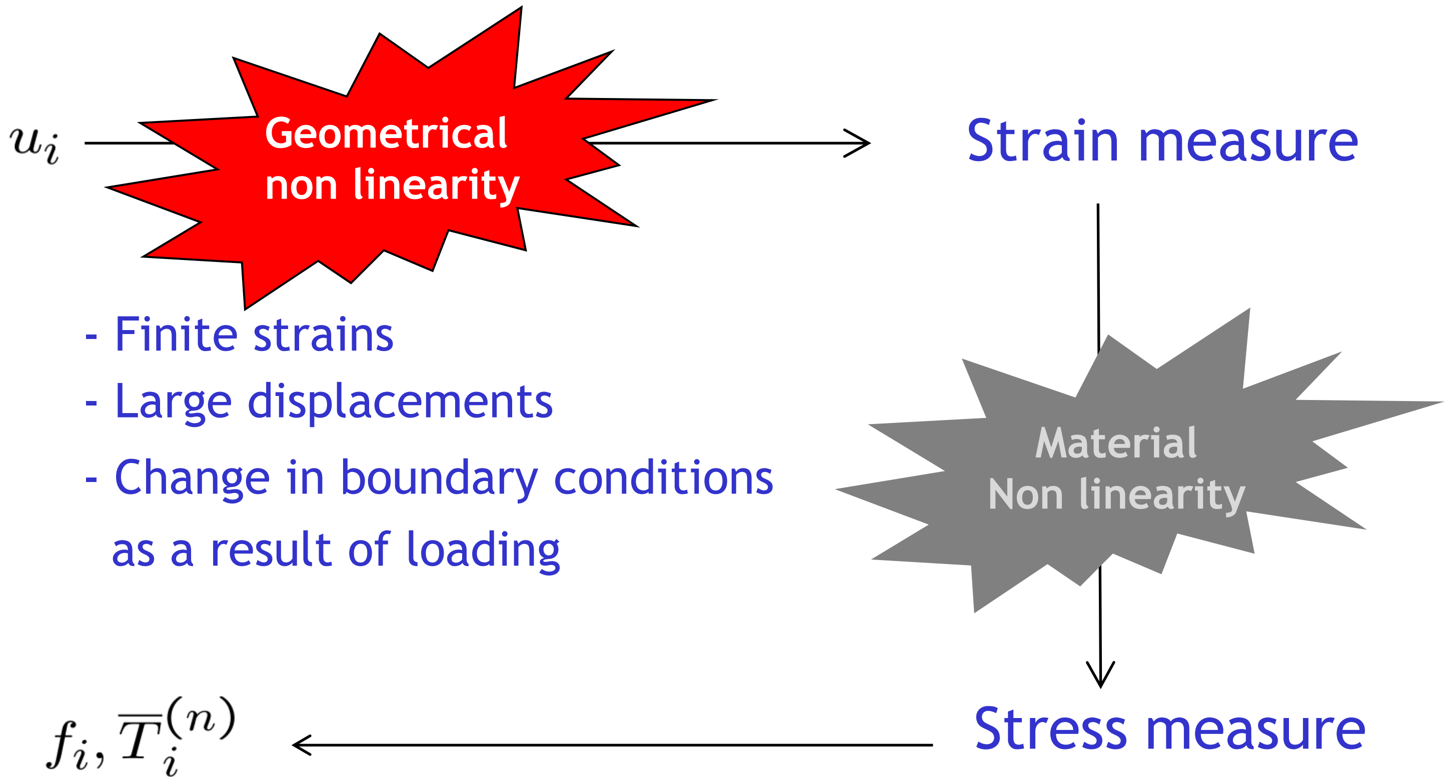
2.1. Geometrical nonlinearities

Inspired and adapted from the 'Nonlinear Modeling of Structures' course of Prof. Thierry J. Massart at the ULB

Definition

Cause of a non proportionnality between applied forces and Resulting displacements

Sources



- Finite strains
- Large displacements
- Change in boundary conditions as a result of loading

Outline

Objective (nonlinear) strain measures

Stress measures

Principle of virtual work

Principle of virtual work

$$\delta W_{int} = \delta W_{ext}$$

with

$$\delta W_{int} = \int_{vol} (\text{stress}) : \delta(\text{conjugate strain}) d(vol)$$

$$\delta W_{ext} = \int_V \vec{f}.\delta\vec{u}dV + \int_S \vec{p}.\delta\vec{u}dS$$

Choice to be made of the description configuration

Choice of configuration (Initial or deformed volume ?)

Choice of the strain measure (which tensor ?)

Choice of the stress measure (which tensor ?)

Strain - uniaxial example

Uniaxial stretch

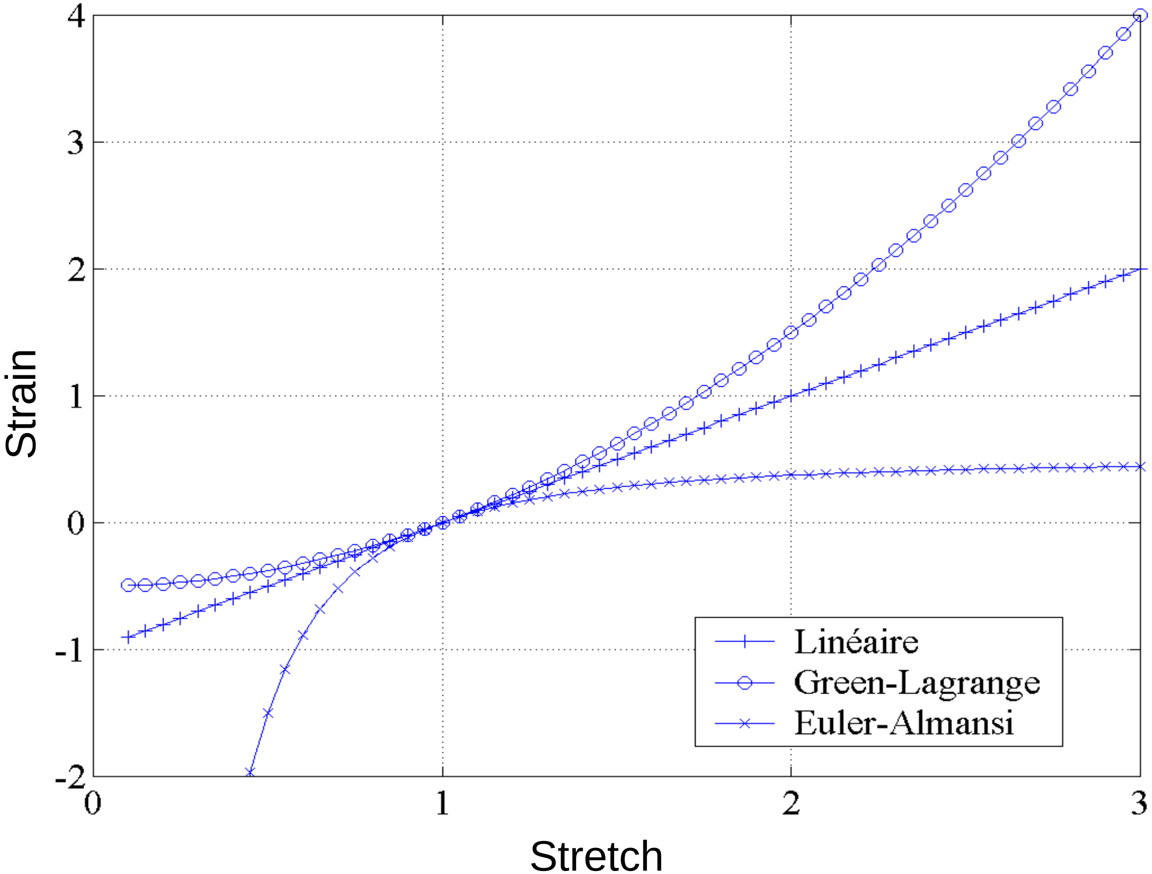
$$\lambda = \frac{l}{l_0}$$

A function $\varepsilon = f\left(\lambda\right)$ is a 'good' strain measure if

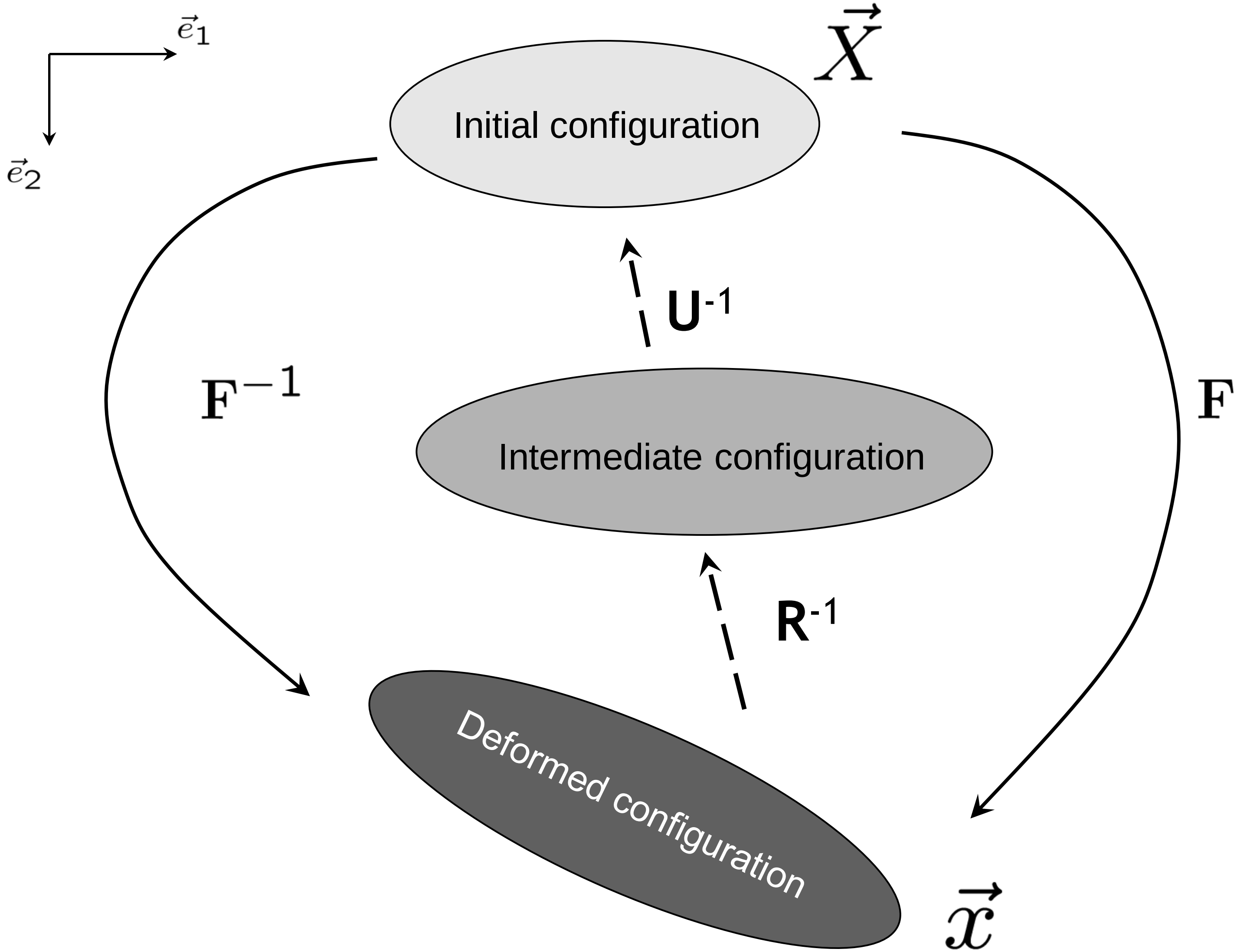
- *f* should vanish when stretch is equal to unity $f\left(\lambda = 1\right) = 0$
- *f* is a monotonic increasing function
- *f* is differentiable function
- *f* is such that $f\left(\lambda\right)|_{\lambda \simeq 1} = \lambda - 1$

Uniaxial example

$$\varepsilon_{lineaire} = \lambda - 1 \qquad \varepsilon_{Green} = \frac{1}{2}\left(\lambda^2 - 1\right) \qquad \varepsilon_{Euler} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)$$



F = RU



$\vec{e}_1$
$\vec{e}_2$
$\vec{X}$
Initial configuration
$\mathbf{U}^{-1}$
$\mathbf{F}^{-1}$
$\mathbf{F}$
Intermediate configuration
$\mathbf{R}^{-1}$
Deformed configuration
$\vec{x}$

Transformation relations

Jacobian of the transformation

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} \quad \text{or} \quad \mathbf{F} = \frac{\partial \vec{x}}{\partial \vec{X}}$$

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} = \frac{\partial X_i + u_i}{\partial X_J} = \delta_{iJ} + \frac{\partial u_i}{\partial X_J}$$

This is often called 'Deformation gradient'

This quantity defines the transformation of an infinitesimal vector

$$dx_i = F_{iJ} dX_J = \frac{\partial x_i}{\partial X_J} dX_J \quad \text{or} \quad d\vec{x} = \mathbf{F} \cdot d\vec{X}$$

Transformation relations

Transformation of an infinitesimal surface element (Nanson)

$$da = J\mathbf{F}^{-T} dA$$

$$\vec{n} da = J\mathbf{F}^{-T} \cdot \vec{N} dA$$

Transformation of an infinitesimal volume

$$dv = J dV = det(\mathbf{F}) dV$$

Multiaxial strain measures

The quantity $\mathbf{F}$ contains the complete transformation (including the rigid body rotation)

A strain measure should not be rotation sensitive

Polar decomposition of the transformation jacobian

$$\mathbf{F} = \mathbf{R}\ \mathbf{U} \quad \text{with} \quad \begin{array}{l} \mathbf{R}\ \mathbf{R}^T = \mathbf{I} \\ \mathbf{U} = \mathbf{U}^T = \sqrt{\mathbf{F}\ \mathbf{F}^T} \end{array}$$

$\mathbf{R}$ Represents the rigid body rotation

$\mathbf{U}$ Represents the 3D stretch (same role as λ en 1D)

$\rightarrow$ It allows objective strain tensors to be defined

Multiaxial strain measures

Different tensors can be defined

- Biot strain $\mathbf{E}^B = \mathbf{U} - \mathbf{I}$
- Logarithmic strain $\mathbf{E}^N = \ln \mathbf{U}$
- Euler strain tensor $\mathbf{E}^E = \frac{1}{2}\left(\mathbf{I} - \mathbf{U}^{-2}\right)$
- Green strain tensor $\mathbf{E}^G = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{F}^T \mathbf{F} - \mathbf{I}\right)$

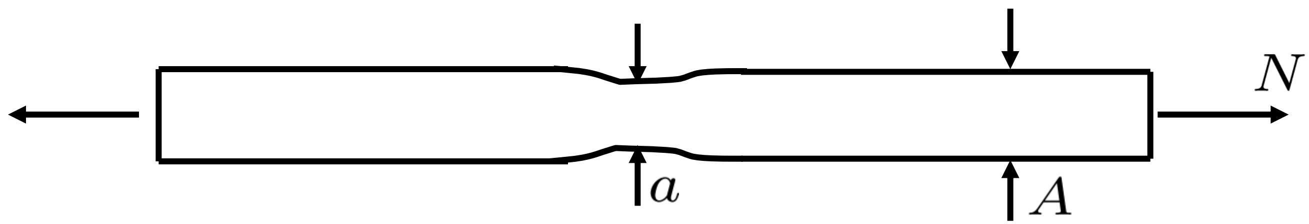
The infinitesimal strain tensor is NOT objective!

$$\varepsilon_{jk} = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_j} + \frac{\partial u_j}{\partial X_k}\right)$$

Stress - uniaxial example

The stress measure can also be dependent on the configuration !

Example: Consider a tensile test with necking



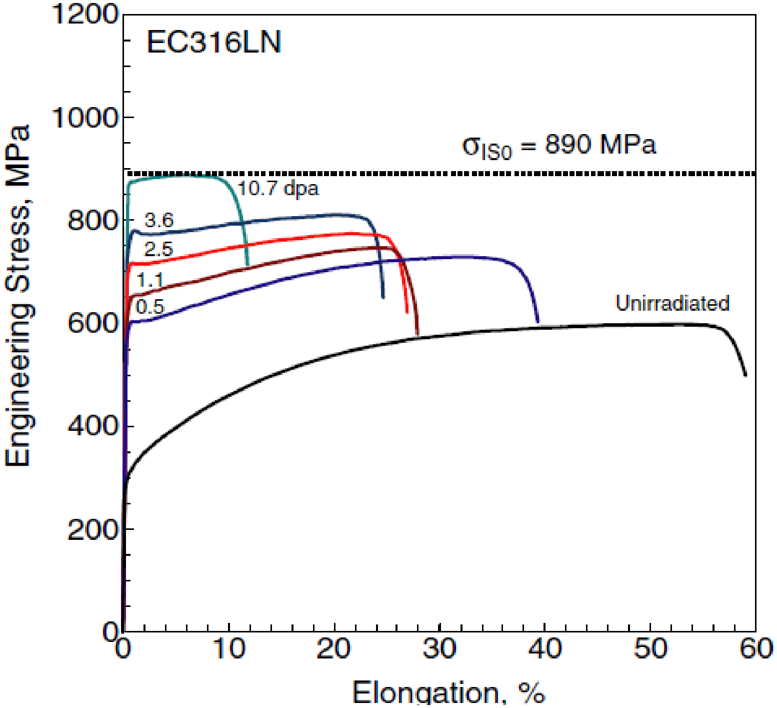
- Initial configuration: $\sigma = \frac{N}{A}$ Nominal stress

- Deformed configuration $\sigma = \frac{N}{a}$ True stress

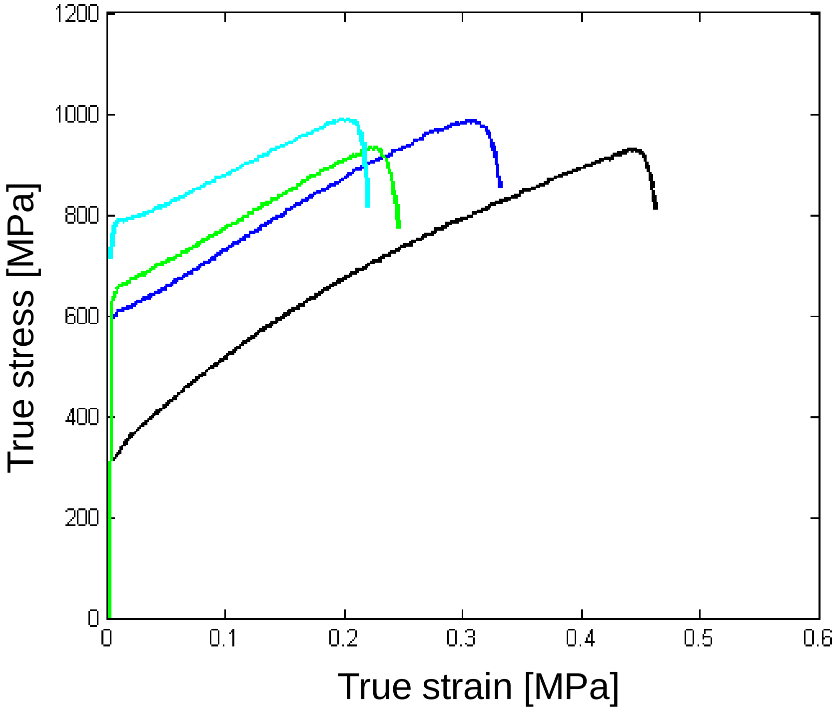
$\rightarrow$ One should take into account the configuration used in the description in the definition of the stress measure !!!

Uniaxial example

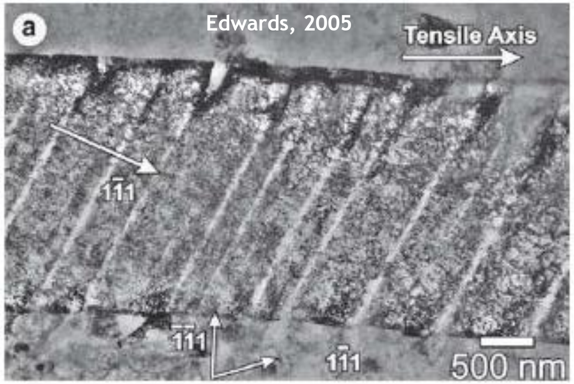
Byun & Farrell, 2004



A. Pajot



Edwards, 2005



Stress vectors

True stress vector $\vec{t} = \lim_{\Delta a \to 0} \frac{\Delta \vec{f}}{\Delta a}$

'Nominal' stress vector $\vec{T}$

True stress vector relative to the initial area

$$\vec{t} da = \vec{T} dA \rightarrow \vec{T} = \vec{t} \frac{da}{dA}$$

Back transformed nominal stress vector $\vec{T}'$

The deformation gradient allows to transform from the initial to the deformed configuration

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} \quad \text{or} \quad \mathbf{F} = \frac{\partial \vec{x}}{\partial \vec{X}}$$ non symmetric !

Back transformed nominal stress vector $\vec{T}' = \mathbf{F}^{-1} . \vec{T}$

Cauchy Stress

True force vector per unit DEFORMED area

(~ generalisation of the uniaxial true stress)

Using the definition of the stress tensor σ_{ij}

$$\vec{t}^{(i)} = \sigma_{ij}\vec{e}^{\,j}$$

Expressing the stress vector as a function of the tensor components

$$\vec{t} = n_i \sigma_{ij} \vec{e}^{\,j} \qquad \text{or} \qquad \vec{t} = \boldsymbol{\sigma}.\vec{n}$$

It is related to areas normal in the deformed configuration

It is symmetric (both indices related to the deformed system)

All components non zero even in uniaxial tension if large displacements

Piola-Kirchhoff 1 Stress

Defined with respect to initial normals
Per unit initial (undeformed) area
(~generalisation of 1D nominal stress)

Through a relation similar to the Cauchy stress definition

$$\vec{T} = P_{iJ} N_J \vec{e}^{\,i} \qquad \text{or} \qquad \vec{T} = \mathbf{P}.\vec{N}$$

where $\vec{N}$ is the normal vector to the facet in the initial configuration

Relates forces in the deformed configuration to initial non deformed normal areas
NOT symmetric
Difficult to interprete physically
Its transposed is sometimes called the nominal stress tensor

Piola-Kirchhoff 2 Stress

Symmetric generalisation of the 1D nominal stress

Based on the back transformed nominal stress vector

Decomposed similarly to the previous tensors

$$\vec{T}' = \mathbf{F}^{-1}.\vec{T} = N_I S_{IJ} \vec{e}^{\,j}$$

Related to initial non deformed normal areas

Symmetric (thanks to the use of $\mathbf{F}^{-1}$)

Difficult to interprete physically

Relations between stress tensors

Definitions

$$\vec{n} \cdot \boldsymbol{\sigma} da = d\vec{f} = \vec{t} da$$

$$\vec{N} \cdot \mathbf{P}^T dA = d\vec{f} = \vec{T} dA$$

$$\vec{N} \cdot \mathbf{S} dA = \mathbf{F}^{-1} \cdot d\vec{f} = \mathbf{F}^{-1} \cdot \vec{T} dA$$

Nanson's relation

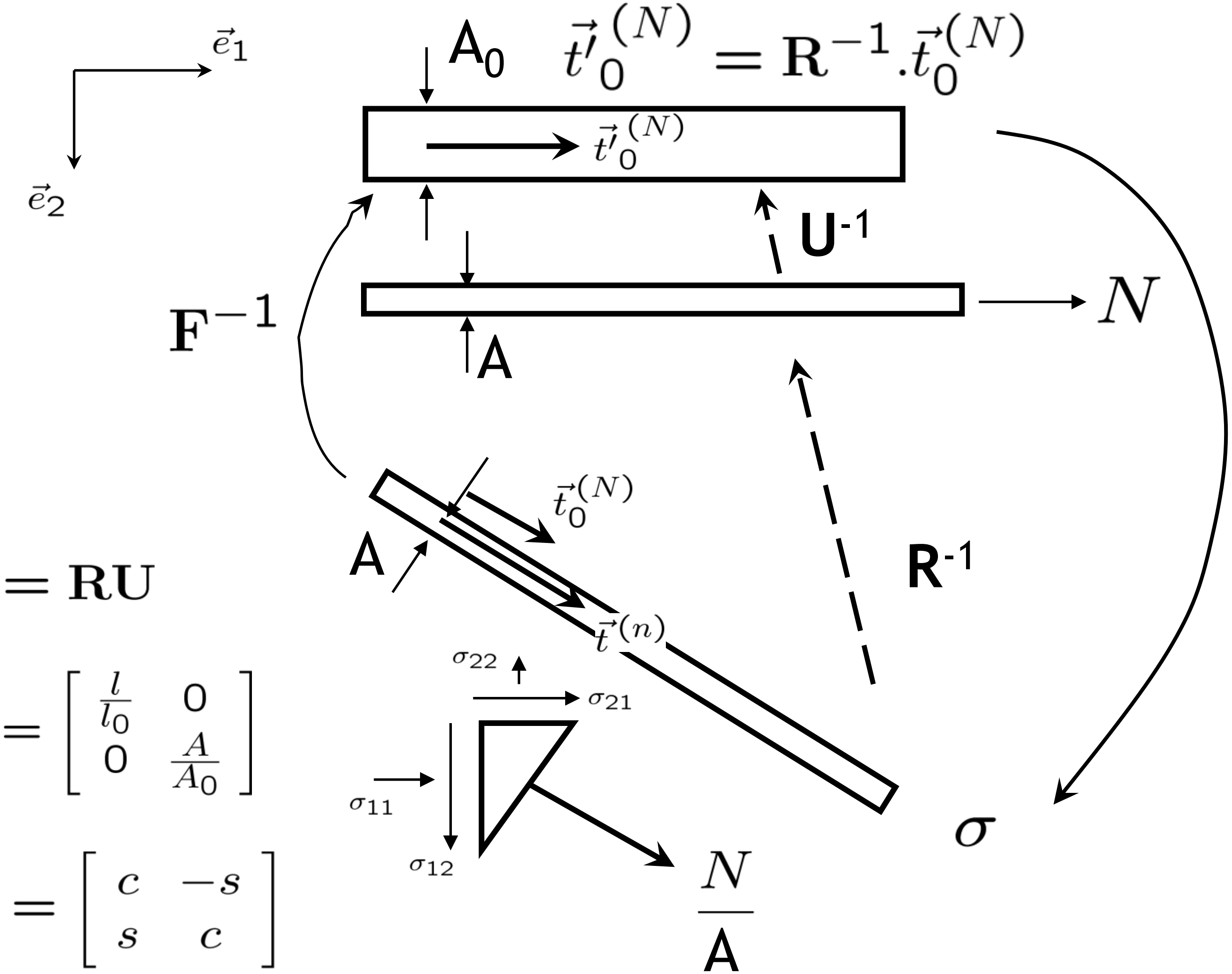
$$\vec{n} da = J \vec{N} \cdot \mathbf{F}^{-1} dA$$

$$\Rightarrow \begin{cases} \mathbf{P}^T & = & J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \\ \mathbf{S} & = & \mathbf{P}^T \cdot \mathbf{F}^{-T} & = & \mathbf{F}^{-1} \cdot \mathbf{P} \\ \boldsymbol{\sigma} & = & J^{-1}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T \end{cases}$$

with $J = det(\mathbf{F})$

Interpretation of stress tensors

In the case of large displacements



$$\mathbf{F} = \mathbf{R}\mathbf{U}$$

$$\mathbf{U} = \begin{bmatrix} \frac{l}{l_0} & 0 \\ 0 & \frac{A}{A_0} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$\boldsymbol{\sigma} = \frac{N}{A} \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

$$\mathbf{P} = \frac{N}{A_0} \begin{bmatrix} c & 0 \\ s & 0 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \frac{N}{A_0}\frac{l_0}{l} & 0 \\ 0 & 0 \end{bmatrix}$$

Remarks

The only ‘physical’ is the Cauchy stress !

In the sense that this is the only one that an be compared to a material strength limit

The Cauchy and PK2 stresses are symmetric

The PK1 tensor is non symmetric !

As this is the case for all tensors which have a ‘leg’ (an index) in each configuration (initial and deformed)

The different stress tensors CANNOT be arbitrarily associated to Any strain tensor to represent properly the internal work !

In the case of material nonlinearity an objective stress rate has to be used

Principle of virtual work

$$\delta W_{int} = \delta W_{ext}$$

with

$$\delta W_{int} = \int_{vol} \text{(stress)} : \delta\text{(conjugate strain)} d(vol)$$

$$\delta W_{ext} = \int_V \vec{f}.\delta\vec{u}dV + \int_S \vec{p}.\delta\vec{u}dS$$

Conjugate quantities

By definition, two stress and strain quantities are conjugate if their internal product integrated on the proper configuration gives the correct internal work

PVW with Cauchy stress

Deformed configuration

$$\delta W_{int} = \int_v \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \, d(v) = \int_v \sigma_{ij} \delta \varepsilon_{ij} \, d(v)$$

with $$\delta \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right)$$

Initial configuration

$$\delta W_{int} = \int_v \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \, d(v) = \int_V (J \boldsymbol{\sigma}) : \delta \boldsymbol{\varepsilon} \, d(V)$$

with $\boldsymbol{\tau} = J \boldsymbol{\sigma}$ called the Kirchhoff stress

PVW with PK1 stress

Initial configuration

$$\delta W_{int} = \int_V \mathbf{P} : \delta \mathbf{g} \, d(V) = \int_V P_{ij} \delta g_{ij} \, d(V)$$

with $$\delta g_{ij} = \frac{\partial \delta u_i}{\partial X_j} = \delta \left(\frac{\partial u_i}{\partial X_j} \right)$$

$$\delta \mathbf{g} = \delta \mathbf{F}$$

PVW with PK2 stress

Initial configuration

$$\delta W_{int} = \int_V \mathbf{S} : \delta\mathbf{E}\, d(V) = \int_V S_{ij} \delta E_{ij}\, d(V)$$

$$\delta E_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial X_j} + \frac{\partial \delta u_j}{\partial X_i} + \frac{\partial \delta u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} + \frac{\partial u_k}{\partial X_i}\frac{\partial \delta u_k}{\partial X_j}\right)$$

which is the Green strain

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

$$\mathbf{E}^G = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{F}^T\,\mathbf{F} - \mathbf{I}\right)$$

Internal work expressions

3 expressions of the internal work variations are thus available

$$w = \int_v \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \; dv$$

$$w = \int_V \mathbf{P} : \delta \mathbf{F} \; dV$$

$$w = \int_V \mathbf{S} : \delta \mathbf{E} \; dV$$

The expressions of the external work may also depend on the chosen configuration (in case of follower forces – think of a balloon inflating)