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Introduction to nonlinear finite element modeling

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1.2. Reminder of 'Prerequisites'

Inspired and adapted from the 'Nonlinear Modeling of Structures' course of Prof. Thierry J. Massart at the ULB

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Outline

Universal laws

Need for constitutive equations

Elasticity - Hooke's law

FEM

Newton-Raphson iterative solution

Universal laws

Mass conservation

→ The mass included in the material volume should be invariant

$$M^{\cdot} = \left[\int_V \rho dV\right]^{\cdot} = 0 \qquad \text{or} \qquad \partial_0 \rho + \partial_i (\rho v_i) = 0$$

Linear momentum conservation

$$\int_V \rho v_i^{\cdot}\, dV = \int_V f_i\, dV + \oint_S T_i^{(n)} dS \qquad \text{or} \qquad \sigma_{ij,j} + f_i = \rho v_i^{\cdot}$$

Conservation of moment of linear momentum

$$\int_V \rho \delta_{ijk} x_j v_k^{\cdot} dV = \int_V \delta_{ijk} x_j f_k dV + \oint_S \delta_{ijk} x_j T_k^{(n)} dS \qquad \text{or} \qquad \sigma_{[ij]} = 0$$

These law are valid whatever the material behavior

Need for constitutive laws

Denumbering the number of unknowns and universal laws

$\rightarrow$ General case (dynamics, 3D) but without thermal exchanges, …

Equations	How much	Unknowns	Number of unkowns
Continuity $\partial_0 \rho + \partial_i (\rho v_i) = 0$	1	ρ, v_i	4
Linear momentum $\sigma_{ij,j} + f_i = \rho \dot{v}_i$	3	σ_{ij}	9
Moment of lin. Mom. $\sigma_{ij,j} + f_i = \rho \dot{v}_i$	3	$-$	3
	7		13

$\rightarrow$ So there's a need to postulate 6 equations specific to the material

$$\sigma_{ij} = f\left(\text{defo}_{kl}\right)$$

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Elasticity - Isotropic Hooke's law

Linear elastic behaviour

$\rightarrow$ Use of infinitesimal strains (small displacements and strains)

$\rightarrow$ There exists a neutral state

σ_{ij} Must be a linear homogeneous function of ε_{kl}

$\sigma_{ij} = H_{ijkl}\varepsilon_{kl} \longrightarrow$ 81 material parameters !

$\rightarrow$ Symmetry of σ_{ij} and ε_{kl}

$H_{ijkl} = H_{(ij)(kl)} \longrightarrow$ 36 material parameters!

$\rightarrow$ In case of isotropy, only 2 material constants enter H_{ijkl}

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}$$

Lamé constants

Young's modulus

Poisson's coefficient

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right]$$

Strong form of the equilibrium problem

Weak form of the equilibrium problem

Matrix notation

Discretised form of the equilibrium problem

Isoparametric formulations and numerical integration

Equilibrium problem - strong form

Equations to be solved

Volume translational equilibrium	$\sigma_{ij,j} + f_i = 0$
Rotational equilibrium	$\sigma_{[ij]} = 0$
Displacement boundary conditions	$u_i = \overline{u}_i \text{ sur } S_u$
Surface equilibrium	$\overline{T}_i^{(n)} = \sigma_{ij} n_j \text{ sur } S_T$
Strain-displacement relationship	$\text{defo}_{ij} = f_{NL}\left(u_{i,j}\right)$
Constitutive equations	$\sigma_{ij} = g_{NL}\,(\text{defo})$

Equilibrium problem - weak form

Equation to be satisfied under weak form

Find the displacement field u_i verifying

$$\sigma_{ij,j} + f_i = 0$$
$$\sigma_{[ij]} = 0$$
$$\overline{T}_i^{(n)} = \sigma_{ij} n_j \text{ sur } S_T$$

'in a weak sense'

$$u_i = \overline{u}_i \text{ sur } S_u$$
$$\text{defo}_{ij} = f_{NL}\left(u_{i,j}\right)$$
$$\sigma_{ij} = g_{NL}\,(\text{defo})$$

'in a strong sense'

$\rightarrow$ There's a choice to be made to satisfy part of the set of equations in a strong sense (another choice could be made)

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Equilibrium problem - weak form

Weak form of equilibrium

Among all the displacement fields u_i satisfying $u_i = \overline{u}_i$ on S_u

find the one which satisfies

$$\int_V w_i \left(\sigma_{ij,j} + f_i\right) dV + \int_V w_{[i,j]} \sigma_{ij} dV + \int_{S_t} w_i \left(\overline{T}_i^{(n)} - \sigma_{ij} n_j\right) dS = 0$$

$\forall w_i$ such that $w_i = 0$ on S_u

This choice of w_i drops the (unknown) reactions as $\int_{S_u} w_i \sigma_i j n_j dS = 0$

Integrating the first term by parts

$$\int_V \sigma_{ij} w_{(i,j)} dV = \int_V w_i f_i dV + \int_{S_t} w_i \overline{T}_i^{(n)} dS$$

$\forall w_i$ Such that $w_i = 0$ on S_u

with $u_i = \overline{u}_i$ on S_u

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Equilibrium problem - weak form

Weak form - energetic interpretation

Among the displacement fields u_i satisfying $u_i = \overline{u}_i$ on S_u

Find the one that satisfies

$$\underbrace{\int_V \sigma_{ij} w_{(i,j)} dV}_{W_{int}} = \underbrace{\int_V w_i f_i dV + \int_{S_t} w_i \overline{T}_i^{(n)} dS}_{W_{ext}}$$

$\forall w_i$ Virtual displacement such that $w_i = 0$ on S_u

Remarks

The solutions of the weak and strong forms are identical

For a (large) number of problems, the weak form can be reformulated as a minimisation problem (including for some non linear problems!)

Matrix notation

Matrices

$[X]$ Matrix with 2 dimensions

$\{X\}$ Column matrix

$\langle X \rangle = \{X\}^t$ Line matrix

Stresses, strains, …

$$\langle \sigma \rangle = \left\langle \begin{array}{cccccc} \sigma_x & \sigma_y & \sigma_z & \sigma_{xy} & \sigma_{xz} & \sigma_{yz} \end{array} \right\rangle$$

$$\langle \varepsilon \rangle = \left\langle \begin{array}{cccccc} \varepsilon_x & \varepsilon_y & \varepsilon_z & \varepsilon_{xy} & \varepsilon_{xz} & \varepsilon_{yz} \end{array} \right\rangle$$

$$\langle u \rangle = \left\langle \begin{array}{ccc} u_x & u_y & u_z \end{array} \right\rangle \qquad \left\langle \overline{T}^{(n)} \right\rangle = \left\langle \begin{array}{ccc} \overline{T}_x & \overline{T}_y & \overline{T}_z \end{array} \right\rangle$$

$$\langle f \rangle = \left\langle \begin{array}{ccc} f_x & f_y & f_z \end{array} \right\rangle$$

Internal and external works

$$\int_V \sigma_{ij} \varepsilon_{ij} dV = \int_V \langle \varepsilon \rangle \{\sigma\} \, dV = \int_V \langle \sigma \rangle \{\varepsilon\} \, dV$$

$$\int_V w_i f_i dV = \int_V \langle w \rangle \{f\} \, dV = \dots$$

Approximation of the weak form

One should test ALL fields u_i satisfying $u_i = \overline{u}_i$ on S_u
w_i $w_i = 0$ on S_u

We choose to test only some subspaces described by discrete unknow values of the method u_i and w_i at certain nodes and interpolation functions $N_i(x)$ with a limited support

This choice introduces an approximation!

$$u^h(x) = \sum_i N^{(i)}(x) u^{(i)} \qquad \left\{u^h(x)\right\} = [N(x)]\{q\}$$

$$w^h(x) = \sum_i N^{(i)}(x) w^{(i)} \qquad \left\{w^h(x)\right\} = [N(x)]\{d\}$$

Strain-displacement relationship in strong form

$$\left\{\varepsilon^h\left(u^h\right)\right\} = [D][N(x)]\{q\} = [B(x)]\{q\}$$

$$\left\{\varepsilon^h\left(w^h\right)\right\} = [D][N(x)]\{d\} = [B(x)]\{d\}$$

Discretisation of the weak form

Among all the displacement fields u^h such that $u^h = \overline{u}$ on S_u

Find the one that satisfies

$$\int_V \left\langle \varepsilon^h\left(w^h\right)\right\rangle \left\{\sigma^h\left(u^h\right)\right\} dV = \int_V \left\langle w^h\right\rangle \{f\}\, dV + \int_{S_t} \left\langle w^h\right\rangle \left\{\overline{T}\right\} dS$$

$\forall w^h$ virtual such that $w^h = 0$ on S_u

Substituting the discrete fields, and imposing equality $\forall \{d\}$

$$\underbrace{\int_V [B]^t \left\{\sigma^h\left(u^h\right)\right\} dV}_{\{f_{int}\}} = \underbrace{\int_V [N]^t \{f\}\, dV + \int_{S_t} [N]^t \left\{\overline{T}\right\} dS}_{\{f_{ext}\}}$$

Remarks

- This is a general format that does depend on the material law
- Computing $\left\{\sigma^h\left(u^h\right)\right\}$ as a fonction of $\{q\}$ requires such a law

Linear elastic case

Introduction of Hooke's law (in a strong sense !)

$$\left\{\sigma^h\left(u^h\right)\right\} = [H]\left\{\varepsilon^h\left(u^h\right)\right\} = [H]\,[B]\,\{q\}$$

Expression of the internal forces explicitly in terms of $\{q\}$

$$\int_V [B]^t \left\{\sigma^h\left(u^h\right)\right\} dV = \underbrace{\left[\int_V [B]^t\,[H]\,[B]\,dV\right]}_{[K]} \{q\}$$

Discretised equilibrium equations $\quad [K]\,\{q\} = \{f_{ext}\}$

For other material laws

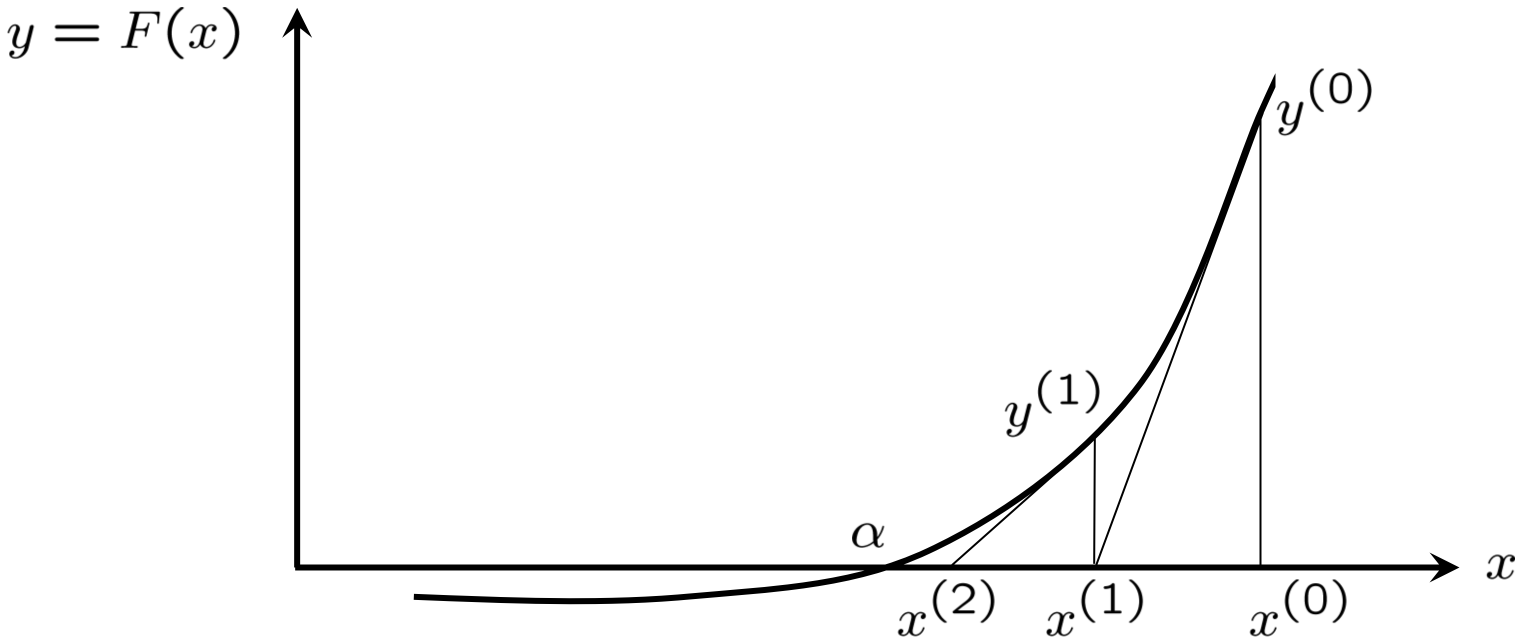
More complex relationship between $\left\{\sigma^h\left(u^h\right)\right\}$ and $\{q\}$

Relationship to be satisfied in a strong sense

Such a relation can potentially be different at different points in an element

Principle

Find a new approximation based on an initial value and the slope of the function at this point



Newton-Raphson

Iterative scheme

Re-write $F(x) = 0$ under the form $x = f(x)$

Construct a series of successive approximations $x^{(1)} = f(x^{(0)})$

$$x^{(k)} = f(x^{(k-1)})$$

Newton-Raphson approximation

$$x^{(1)} = x^{(0)} - \frac{F(x^{(0)})}{F'(x^{(0)})}$$

$$x^{(k+1)} = x^{(k)} - \frac{F(x^{(k)})}{F'(x^{(k)})}$$

Requires the knowledge of the function derivative

Quadratic local convergence, <u>if the derivative is right</u>

This last point is CRUCIAL for a proper convergence!

Interpretation from a series development

Assume a first approximation is available $x^{(k)}$

Express the value of the function as a first order development

$$F(x^{(k+1)}) = F(x^{(k)}) + F'(x^{(k)})(x^{(k+1)} - x^{(k)}) + \cancel{\frac{F''(x^{(k)})}{2!}(x^{(k+1)} - x^{(k)})^2 + \ldots}$$

If this new value has to vanish (to find the root)

$$F(x^{(k+1)}) = 0$$

Using the first order development, a new approximation is

$$F(x^{(k)}) + F'(x^{(k)})(x^{(k+1)} - x^{(k)}) \approx 0$$

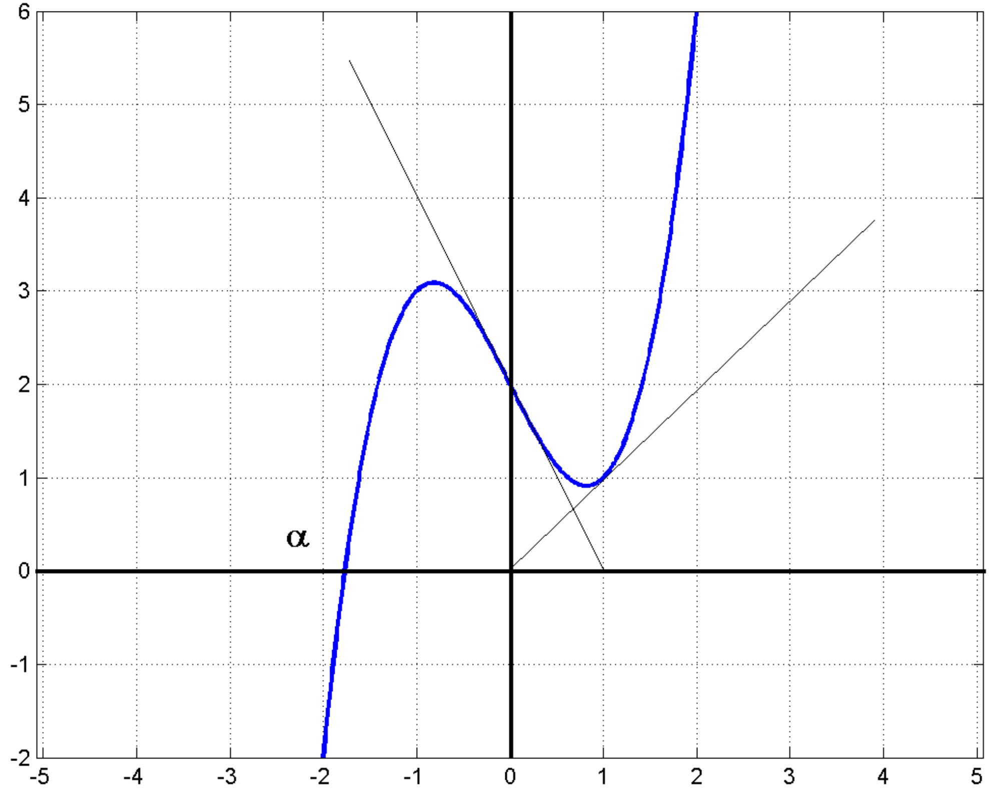
$$x^{(k+1)} = x^{(k)} - \frac{F(x^{(k)})}{F'(x^{(k)})}$$

Potential shortcomings

Vanishing derivatives

Deadlocks between particular points

$$F(x) = x^3 - 2x + 2$$



Systems of NL equations

For the system of equations

$$F_1(x_1, ..., x_n) = 0$$
$$\cdots$$
$$F_n(x_1, ..., x_n) = 0$$

The iterative scheme becomes

Initial approximation $\left\{x^{(0)}\right\} = \left\{x_1^{(0)}, ..., x_n^{(0)}\right\}^T$

A new approximation is found by solving

$$\left[J_F\left(\left\{x^{(k)}\right\}\right)\right]\left(\left\{x^{(k+1)}\right\} - \left\{x^{(k)}\right\}\right) = -\left\{F\left(\left\{x^{(k)}\right\}\right)\right\}$$

Jacobian matrix